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HIDDEN sl(2,R) SYMMETRY IN 2D CFTs AND THE WAVE FUNCTION OF 3D QUANTUM GRAVITY

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Abstract

We show that all two-dimensional conformal field theories possess a hidden $sl(2,R)$ affine symmetry. More precisely, we add appropriate ghost fields to an arbitrary CFT, and we use them to construct the currents of $sl(2,R)$. We then define a BRST operator whose cohomology defines a physical subspace where the extended theory coincides with the original CFT. We use the $sl(2,R)$ algebra to construct candidate wave functions for 3-d quantum gravity coupled to matter, and we discuss their viability.

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1 Introduction, Motivations (and Pipe Dreams)

Quantum gravity is still a mysterious theory, despite the enormous progress made in the last decades toward its understanding, mostly thanks to string theory. Even some of the most basic questions are still unanswered. For instance, we do not have a clear understanding of what the fundamental, non-redundant degrees of freedom of quantum gravity are. One bold attempt to define them is the holographic principle [1, 2]. It states that the true degrees of freedom of quantum gravity in a region V of a d-dimensional space can be thought of as describing a $d-1$-dimensional field theory, living at an appropriately defined boundary of the region V. The initial motivation for this idea is the famous Bekenstein bound [3] on the entropy of a black hole: $S = A/4G_N$ (A = area of the black-hole horizon, G_N = Newton's constant in d dimensions).

When the $d+1$-dimensional space-time is Anti de Sitter (AdS_{d+1}), this correspondence can be made more concrete. In this case, the space-time boundary is a time-like surface; it is conformal to the Einstein Static Universe, $S_{d-1} \times R$. In this case, the conjecture is that a consistent quantum gravity in an asymptotically AdS_{d+1} space is equivalent (dual) to a *local, non-gravitational* field theory, living on $S_{d-1} \times R$. The theory has a conformal fixed point in the ultraviolet (UV) [4]. The equivalent theories are dual to each other, in the sense that when one of them, say the field theory on $S_{d-1} \times R$ is strongly coupled, the other one, quantum gravity on AdS_{d+1}, is weakly coupled (e.g. semi-classical).

Our description of the holographic duality has omitted many details. We should selectively add them as needed in our discussion. The first one is that the space gravity lives in, is not just AdS_{d+1}, but, generically, a 10- or 11-dimensional manifold whose metric is a warped product of AdS_{d+1} times some compact space. The first example of holographic duality, for instance, was between Type IIB superstring theory on $AdS_5 \times S_5$, and a 4-d supersymmetric Yang-Mills theory with 16 supercharges and gauge group $SU(N)$ [5]. The coupling constant of the $SU(N)$ super Yang-Mills theory (SYM) is the 't Hooft coupling g^2N. The dual meaning of this parameter is the curvature radius l of AdS_5, in units of the string length l_S: $l \sim (g^2N)^{1/4} l_S$ [5]. Hence, the semi-classical regime for gravity $l \gg l_S$, holds precisely when the SYM theory is strongly coupled. In this example, the $AdS_5 \times S_5$ background is a near-horizon geometry of N parallel D3 branes.

Other examples of AdS holography have been worked out in the literature. A particularly interesting one gives origin to an AdS_3/CFT_2 duality [6, 7]. In this case, one starts with a configuration of Q_1 D1 branes and Q_5 D5 branes of Type IIB superstring on $R^{(5,1)} \times M^4$, with

M^4 either T^4 or K_3. The near-horizon geometry of this configuration is $AdS_3 \times S_3 \times M^4$. The holographic dual is the infrared limit of a 2-dimensional conformal field theory, which has central charge $c = Q_1 Q_5$ [up to $O(1)$ corrections] and is believed to be a deformation of the orbifold sigma models living on the symmetric product of c copies of M^4. The curvature radius of AdS_3 is $l \sim (Q_1 Q_5)^{1/4} G_N$, where G_N is the 3-dimensional Newton constant.

More generally, one can conjecture that any *consistent* quantum gravity on AdS_3 is described by some conformal field theory. One important clue to this conjecture is that the algebra of asymptotic isometries in AdS_3 gravity is the Virasoro algebra with central charge $c = 3l/2G_N$ [8].

A special case is pure AdS_3 gravity. This theory does not propagate any local degrees of freedom, so, by construction, it has only a boundary dynamics. We shall explain in details in Section 2 how to rewrite pure (2+1)-dimensional gravity as a Chern-Simons (CS) theory with gauge group $SL(2, R) \times SL(2, R)$. Here we only remark that any CS theory is "hyper-holographic." By this we mean the following.

1.1 Hyperholography

Euclidean AdS_{d+1} space is topologically a $d+1$-dimensional ball, B_{d+1}. The space can be foliated by spheres S_d, with coordinates x^μ, together with a radial coordinate r. Near the boundary $r = 0$, the line element is

$$ds^2 \approx \frac{l^2}{r^2}[dr^2 + g_{\mu\nu}(r,x)dx^\mu dx^\nu], \qquad g_{\mu\nu}(r,x) = g^0_{\mu\nu}(x) + O(r^2). \tag{1}$$

In radial quantization, r plays the role of time: to describe quantum gravity, one gives the wave function "of the universe" at fixed r, and dynamics is radial evolution. The AdS/CFT duality becomes the statement that the wave function of the universe at $r = \epsilon \ll l$ is the partition function of a CFT regularized with cutoff ϵ [9]:

$$\Psi(\epsilon) = Z^\epsilon_{CFT}. \tag{2}$$

The cutoff ϵ must be introduced because the partition function of the CFT, Z^ϵ_{CFT}, may (does) contain divergent contact terms. More concretely, if we make explicit the functional dependence of Ψ_ϵ on the metric we have

$$\Psi[\epsilon, g_{\mu\nu}] = \exp\{-\int d^d x [A\epsilon^{-d} + B\epsilon^{2-d} R(g) + ...] - \Gamma^F_\epsilon[g]\}. \tag{3}$$

The ellipsis denote divergent, *local* functions of the metric, A, B are constants, and $\Gamma^F_\epsilon(g)$ is a (non-local) functional, finite in the limit $\epsilon \to 0$.

CS theories obey an identity stronger than Eq. (3). In that case, given any 3-dimensional space M with boundary $\Sigma = \partial M$, one has [10]

$$\Psi_{CS,k}[\Sigma] = Z^{\Sigma}_{WZW,k}. \tag{4}$$

In words: on *any* surface Σ, the wave function of the CS theory with gauge group G and coupling constant k *is* the partition function of the level-k chiral WZW model of the group G. This definition can be easily modified so that it makes sense also when the surface Σ has a boundary. In this case, it is no longer true that $\Sigma = \partial M$, rather: $\Sigma \subset \Sigma' = \partial M$. Indeed, Eq. (4) makes sense for a larger class of 2-dimensional CFTs, as we will argue now.

1.2 A Wave Function of the Universe?

The identity Eq. (4) holds because both sides of the equation obey the same defining functional equation: the LHS obeys a quantum Gauss law, while the RHS obeys the Ward identities of the affine-Lie algebra based on the group G [11]. Now, the Gauss law is the defining equation for the wave function, so we may ask whether one can get a reasonable Gauss law [1] from other 2-d CFT, besides the chiral WZW model. Consider in particular a generic 2-d CFT possessing holomorphic dimension-1 currents $J_a(z)$, which generate the affine-Lie algebra $\hat{g}$ based on the group G. They obey the well-known OPE

$$J_a(z)J_b(w) = \frac{k\delta_{ab}}{(z-w)^2} + \frac{if^c_{ab}}{(z-w)}J_c(z) + \text{regular terms}. \tag{5}$$

The f^c_{ab} are the structure constants of the group, and k is the level of the affine algebra. Consider now the (genus-0) partition function of this model, in the presence of a background gauge field $A^a_{\bar{z}}$. This is also the generating functional for the Green's functions of the currents J_a:

$$Z[A_{\bar{z}}] \equiv \langle 0| \exp \int d^2z J_a A^a_{\bar{z}} |0\rangle. \tag{6}$$

On the background $A^a_{\bar{z}}$, the current is covariantly conserved up to an anomaly

$$(D_{\bar{z}}J)^a = \frac{ik}{2\pi}\partial_z A^a_{\bar{z}}, \qquad D^a_{\bar{z}\,b} = \delta^a_b\partial_{\bar{z}} + f^a_{bc}A^c_{\bar{z}}. \tag{7}$$

Substituting this equation into the definition Eq. (6), we get

$$\left[D_{\bar{z}}\frac{\delta}{\delta A^a_{\bar{z}}(z)} - \frac{ik}{2\pi}\partial_z A^a_{\bar{z}}(z)\right] Z[A_{\bar{z}}] = 0. \tag{8}$$

[1] For gravity, the quantum Gauss law is of course the Wheeler-De Witt equation.

If we interpret $Z[A_{\bar z}]$ as a wave function, and we define the canonically conjugate variable to $A^a_{\bar z}$ by

$$A_{z\,a}(z,\bar z) \equiv -\frac{2\pi i}{k}\frac{\delta}{\delta A^a_{\bar z}(z,\bar z)}, \tag{9}$$

then, Eq. (8) becomes the Gauss law $F^a_{z\bar z}Z[A_{\bar z}]=0$.

An obvious generalization is to insert local operators in Eq. (6)

$$Z_{\mathcal{O}}[A_{\bar z}] = \langle 0|R\prod_i \mathcal{O}^i(z_i)\exp\int d^2z J_a A^a_{\bar z}|0\rangle, \qquad R\text{= radial ordering.} \tag{10}$$

The operators $\mathcal{O}^i$ may have a singular OPE with J^a. When they are primaries of the affine-Lie algebra, the OPE is particularly simple

$$J^a(z)\mathcal{O}^i(w) = \frac{1}{z-w}T^{a\,i}_j\mathcal{O}^j. \tag{11}$$

The matrices $T^{a\,i}_j$ define a representation of the group G.

The anomalous Ward identity now gives the Gauss law with external point-like charges

$$F^a_{z\bar z}Z_{\mathcal{O}}[A_{\bar z}] = \sum_i k\delta^2(z-z_i)\langle 0|R\prod_{l\neq i}\mathcal{O}^l(z_l)T^{a\,i}_j\mathcal{O}^j(z_i)\exp\int d^2z J_a A^a_{\bar z}|0\rangle. \tag{12}$$

More generally, one can define a wave function by considering the CFT on a disk D, with boundary state $|B\rangle$ at ∂D.

$$Z_{\mathcal{O},B}[A_{\bar z}] = \langle 0|R\prod_i \mathcal{O}^i(z_i)\exp\int_D d^2z J_a A^a_{\bar z}|B\rangle. \tag{13}$$

For any point $z\in D-\partial D$, $Z_{\mathcal{O},B}[A_{\bar z}]$ obeys the Gauss law Eq. (12).

Equation (12) is suggestive. If we take it literally, it says that the partition function of *any* CFT possessing an affine-Lie algebra can be reinterpreted as the physical wave function of a CS theory coupled to matter. Notice that the CFT need not be a WZW model. Furthermore, as we will review in Section 2, pure gravity in 3-d is a CS theory with gauge group $SL(2,R)\times SL(2,R)$. So, Eq. (12) or (13) may be used to define the Wheeler-De Witt wave function.

1.3 Plan of the Paper

Our last statement was quite imprecise. There are many subtle points to understand before we can, even tentatively, identify Eq. (13) with the wave function of 3-d AdS gravity. In Section 2 we review the construction of CS 3-d gravity, and its reduction to a boundary WZW model

with constraints. In the course of the review, we address the first subtlety. The problem is that, on manifolds with boundaries, pure 3-d gravity needs an extra constraint on the space of states. Section 2 shows how to modify accordingly the recipe for the physical wave function.

The second problem is whether an affine-Lie symmetry $sl(2,R)$ exists, inside a generic CFT. Better, whether, given an arbitrary CFT, one can extend it in such a way that: a) The extended theory possesses an affine $sl(2,R)$. b) One can define a physical subspace within the extended theory, where it reduces to the original CFT. c) The stress-energy tensor of the extended theory coincides with that of the original theory on physical states. d) The physical subspace is defined by a BRST cohomology. e) Any physical operator of the original CFT can be "dressed" appropriately so that it transforms in a representation of the affine $sl(2,R)$. In Sections 3 to 7 we show that all these conditions can be satisfied.

Section 3 answers the first question. There, a minimal set of auxiliary ghost fields is defined, that allow us to construct the currents of $sl(2,R)$. Our construction is a modification of the well-known free-field realization of the $sl(2,R)$ current algebra. Section 4 uses a known technique for modifying the stress-energy tensor, and consistently implement the extra constraint needed to reduce CS states to physical, 2-d gravity states.

Section 5 introduces the BRST operator that defines the physical space, and shows that the stress-energy tensor of the extended theory equals that of the original theory, up to a BRST-exact term, thereby answering question c). Points b) and d) are answered in Section 6, where the BRST cohomology is proved to coincide with the Hilbert space of the original CFT. Section 7 addresses the last point: it gives a constructive recipe to build irreducible representations of the affine $sl(2,R)$ starting from the Virasoro primaries of the original CFT.

Sections 3 to 7 are the converse of ref. [12]. There, it was shown that an irreducible representation of affine $sl(2,R)$ can be constrained so as to give a single, irreducible representation of the Virasoro algebra (see ref. [13] for the supersymmetric extension of this result). Here, we show how to embed a CFT, i.e. a collection of representations of the Virasoro algebra, into a collection of representations of $sl(2,R)$.

The most serious problem toward using Eq. (13) to define a physical wave function for 3-d gravity arises precisely in the presence of matter. In that case, the Gauss law becomes $F^a_{z\bar{z}}\Psi = \rho^a\Psi$, where ρ^a is the charge density of $SL(2,R)$. The problem is that this charge density must also be the stress-energy tensor of matter. Whether ρ^a gives an acceptable stress-energy tensor is an open problem. This point is discussed more extensively in Section 8, which also contains our conclusions.

Technical material on AdS boundary conditions is confined to Appendix A. Appendix B gives a derivation of the well-known Künneth formula in BRST cohomology, and is included for completeness.

2 $sl(2,R)$ Affine Lie Symmetry in $(2+1)$-Dimensional Gravity

In this Section we review the $sl(2,R)$ affine algebra structure of gravity in (2+1) dimensions with a negative cosmological constant $\Lambda = -1/l^2$, and how AdS_3 boundary conditions determine the constraints on the affine currents, following [14, 15].

2.1 3-d Gravity as CS

Einstein gravity in (2+1) dimensions with a negative cosmological constant can be reformulated in terms of two copies of an $SL(2,R)$ CS gauge theory [16, 17, 18]. With the definitions

$$A^a = \frac{e^a}{l} + \omega^a, \qquad \tilde{A}^a = -\frac{e^a}{l} + \omega^a, \tag{14}$$

the Einstein-Hilbert action becomes

$$S_E \equiv \frac{1}{8\pi G_N} \int \left(e^a \wedge R_a - \frac{1}{6\,l^2}\epsilon_{abc}\, e^a \wedge e^b \wedge e^c \right) = S_{CS,k}[A] - S_{CS,k}[\tilde{A}]. \tag{15}$$

Here $S_{CS,k}[A]$ is the Chern-Simons action with coupling constant k for the $SL(2,R)$ gauge connection $A = A^a t^a$, t^a are the generators of the $sl(2,R)$ Lie algebra in the fundamental representation, and $k = -l/4G_N$. The topological character of the CS action implies that there are no local degrees of freedom in this theory, and the dynamics is given entirely in terms of holonomies of the (flat) gauge connections A and $\tilde{A}$. Things change however if the 3-d manifold on which the theory is defined has a time-like boundary: in this case the CS gauge theory has an infinite number of degrees of freedom, consisting of the values of the gauge fields at the boundary, and the model describing the boundary dynamics possesses an affine-Lie algebra structure, based on the group $SL(2,R)$.

On the other hand, if we consider the metric description of AdS_3 gravity, we must demand that the metric approaches the AdS_3 metric near the boundary. Using coordinates $(r, x^+ = t/l + \phi, x^- = -t/l + \phi)$, this metric reads

$$ds^2 \approx l^2 \left(dr^2 + e^{2r} dx^+ dx^- \right), \qquad r \to \infty. \tag{16}$$

This condition is obeyed only by a restricted class of CS connections; therefore, the theory describing the dynamics of asymptotically AdS_3 spaces, will not possess the full affine-Lie symmetry of the boundary CS, but only a subgroup preserving the boundary conditions Eq. (16). This subgroup, made of two copies of the Virasoro algebra, was found long ago [8] to be the asymptotic symmetry group of AdS_3. One can enforce this restriction on the dynamical degrees of freedom through a mechanism called Hamiltonian reduction. It can be realized by gauging part of the full algebra, so that only a subalgebra connects physically inequivalent states. This mechanism is the main subject of the next Sections. In this Section, we review the constraints that AdS_3 boundary conditions impose on the affine currents.

Consider the bulk CS action[2]

$$S^0_{CS,k}[A] \equiv \frac{k}{4\pi}\mathrm{Tr}\int_M \left(A\wedge dA + \frac{2}{3}A\wedge A\wedge A\right) = \frac{k}{4\pi}\mathrm{Tr}\int_M \epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right), \quad (17)$$

where M is a 3-manifold with coordinates (r,t,ϕ), whose boundary is parametrized by (t,ϕ), or (x^+,x^-). To have a well defined variation, this action needs boundary conditions that fix one of the components of the gauge field A. Different choices of boundary conditions can be implemented by adding appropriate boundary terms to Eq. (17), and demanding that the action is stationary with respect to *all* smooth variations of the fields, even those that do not vanish at the boundary. By adding the boundary term Tr $\int_{\partial M} A_t A_\phi$ to Eq. (17), one brings the action in a "canonical" form, where A_t appears explicitly as a Lagrange multiplier:

$$S^1_{CS,k}[A] \equiv S^0_{CS,k}[A] + \frac{k}{4\pi}\mathrm{Tr}\int_{\partial M} A_t A_\phi = \frac{k}{4\pi}\mathrm{Tr}\int_M \left(A_\phi\partial_t A_r - A_r\partial_t A_\phi + 2A_t F_{r\phi}\right). \quad (18)$$

The variation of this action w.r.t. A_t yields the constraint $F_{r\phi}=0$, while the variation w.r.t. the other components of A gives terms proportional to the equations of motions $F_{tr}=F_{t\phi}=0$, plus the boundary term $(k/4\pi)\mathrm{Tr}\int_{\partial M} A_t\delta A_\phi$. To make it vanish we must require the boundary condition $A_t=0$. However, as we review in Appendix A, the boundary conditions for asymptotically AdS_3 space-times are $A_-=0$, or $A_t=A_\phi$. To enforce them, we add to Eq. (18) an additional boundary term, and use the following definition of the CS action:

$$S_{CS,k}[A] \equiv \frac{k}{4\pi}\mathrm{Tr}\int_M \left(A_\phi\partial_t A_r - A_\rho\partial_t A_\phi + 2A_t F_{r\phi}\right) + \frac{k}{4\pi}\mathrm{Tr}\int_{\partial M} A_\phi^2. \quad (19)$$

The constraint $F_{r\phi}=0$ is solved by the requirement that the space part of the connection is flat. On a disk without punctures this implies

$$A_r = U\partial_r U^{-1}, \qquad A_\phi = U\partial_\phi U^{-1}, \quad (20)$$

[2]To avoid repetitions, we consider here only one of the two CS theories in Eq. (15).

with $U(t, r, \phi)$ an arbitrary element of $SL(2, R)$. Substituting Eq. (20) back into Eq. (19), and integrating by parts, we get an induced action for the group element U, which reads

$$\begin{aligned} S^+[U] &= \frac{k}{4\pi} \mathrm{Tr} \int_{\partial M} \left[\left(U \partial_t U^{-1} \right) \left(U \partial_\phi U^{-1} \right) - \left(U \partial_\phi U^{-1} \right)^2 \right] dt d\phi \\ &+ \frac{k}{12\pi} \mathrm{Tr} \int_M \epsilon^{\mu\nu\rho} \left[\left(U \partial_\mu U^{-1} \right) \left(U \partial_\nu U^{-1} \right) \left(U \partial_\rho U^{-1} \right) \right] d^3x. \end{aligned} \tag{21}$$

This is the chiral WZW action [19, 20] for U. It is 2-dimensional, since it depends only on the boundary value of U: $U(t, \phi)$. It has an affine $SL(2, R)$ symmetry of the form $U \to h(x^+)U$, generated by the right-moving current $J(x^+) = k\, U \partial_\phi U^{-1} = k\, A_\phi = k\, A_+/2$. There is only a right-moving affine Lie symmetry because the boundary condition $A_- = 0$ is preserved only by gauge transformations on A that are independent of x^- at the boundary.

The same derivation can be carried out for the other CS theory, with connection $\tilde{A}$: for this we impose the boundary condition $\tilde{A}_+ = 0$, leading to a chiral WZW model on the boundary, with a left-moving affine-Lie symmetry generated by the current $\tilde{J}(x^-) = k\, \tilde{A}_-/2$. This shows that the boundary degrees of freedom of gravity in (2+1)-dimensions realize two independent chiral affine Lie algebras.

Up to now, we have imposed only a minimal set of boundary conditions on the CS gauge fields, just enough to make the variation of the action well defined. If we want the CS theory to describe gravity in an asymptotically AdS_3 space-time, we need to put further restrictions on the boundary values of A and $\tilde{A}$. To see this, recall that in an asymptotically AdS_3 space-time, the metric near the boundary must reduce to the form given in Eq. (16). More precisely, the metric must have the asymptotic behavior [8]:

$$\begin{aligned} ds^2 &= l^2 \Big[\left(1 + O(e^{-2r}) \right) dr^2 + \left(e^{2r} + O(1) \right) dx^+ dx^- \\ &+ O(1)(dx^+)^2 + O(1)(dx^-)^2 + O(e^{-2r})(dr dx^+ + dr dx^-) \Big]. \end{aligned} \tag{22}$$

As shown in Appendix A, this asymptotic behavior not only requires the boundary condition[3]

$$A_- = 0, \qquad \tilde{A}_+ = 0, \tag{23}$$

but it also constrains the WZW affine currents to satisfy

$$J^- = k, \qquad J^3 = 0; \qquad \tilde{J}^+ = -k, \qquad \tilde{J}^3 = 0, \tag{24}$$

with arbitrary J^+ and $\tilde{J}^-$.

[3] The upper index refers the $sl(2, R)$ Lie algebra, the lower one to space-time coordinates

Clearly this restriction on the boundary values of the fields breaks the affine-Lie symmetry; as we will see shortly, it leaves only a conformal symmetry of the boundary, generated by two independent copies of the Virasoro algebra. This can be shown by imposing $J^- = k$ and $J^3 = 0$ as classical constraints in the boundary theory, and observing that the Dirac brackets of the only remaining field, $J^+(x^+, x^-) = kL(x^+, x^-)$, are precisely those of a Virasoro algebra with central charge $c = -6k$ [15]. Another approach [14, 21] consists in further reducing the action (21) by imposing the constraints *after* having combined the two chiral WZW models in a single non-chiral one, with group element $g^{-1}(x^+)\tilde{g}(x^-)$. As it was shown in ref. [22] the result of this reduction is the Liouville theory, which carries an action of the left and right Virasoro algebras. We will not follow these approaches. Instead, we will impose $J^- = k$ *after* quantization, as a constraint on physical states. In this approach, i.e. if we first quantize and then impose the constraint, the condition $J^- = k$ is enough to get rid of the unphysical degrees of freedom, whereas the condition $J^3 = 0$ is a gauge-fixing.

Until now we have discussed pure gravity, but one expects this discussion to be valid also for gravity coupled to arbitrary sources localized in the bulk. In fact, the Virasoro algebra is an asymptotic isometry of AdS_3 even when matter is added to pure 3-d gravity [23]. This is one of the basis for the strong version of the AdS/CFT correspondence in 2+1 dimensions, which conjectures that any *consistent* theory of quantum gravity in an asymptotically AdS_3 space-time is dual to some 2-d CFT living on the boundary. One may wonder if this result is valid not only for the conformal structure, but also for the affine-Lie structure, i.e. if, for a generic theory of AdS_3 gravity, one can construct affine currents acting on the boundary, which, upon restriction to AdS_3 boundary conditions, reduce to the Virasoro algebra. It would be very hard to check this fact along the lines followed in this Section. Indeed, the CS description is simple only in the case when matter is made of point-like external sources in the bulk. In this case, sources are represented by punctures in the disk, together with their associated gauge-field holonomies. For a general matter QFT coupled to gravity, instead, it is generically impossible to reformulate the model as an ordinary gauge theory: the kinetic terms of the matter involve the inverse of the dreibein, so that the coupling to the CS gauge field is very complicated, and certainly non-minimal.

The presence of the boundary affine-Lie symmetry, instead, follows from a *purely 2-d* result. This is the main point of this paper: we will show in Sections 3 to 7 that all 2-d CFTs possess a "hidden" $sl(2, R)$ affine algebra, namely, that it is always possible to embed the Virasoro algebra in a larger $sl(2, R)$ affine structure, by adding appropriate auxiliary fields. We will also define

a physical subspace –and physical observables– such that the restriction of the new theory to that physical subspace gives back the original theory and the original Virasoro algebra. Before doing that, we make some remarks about how the constraints on the current are imposed on the wave function.

2.2 The Extra Constraint

Before describing the construction of the hidden $sl(2,R)$ algebra, we must show how the extra constraint changes the definition of the wave function Eq. (13). Clearly, nothing changes for closed 2-d manifolds. When the 2-d manifold has a non-empty boundary, instead, the extra constraint imposes a restriction on the boundary state. In the simple case considered in Eq. (13), i.e. a disk with punctures, the constraint on the state $|B\rangle$ is

$$(J^-_n - k\delta_{n,0})|B\rangle = (\tilde{J}^+_n + k\delta_{n,0})|B\rangle = 0, \qquad \forall n, \tag{25}$$

where J^-_n are the modes of the current $J^-(z)$. In later Sections we will recast this constraint in a BRST form $Q_B|B\rangle = 0$. In either form, Eq (25) implies a set of Ward identities on the wave function. Consider for instance the constraint following from $J^- = k$:

$$\langle 0|R \oint_{C=\partial D} \frac{dw}{2\pi i} w^n [J^-(w) - k] \prod_i \mathcal{O}^i(z_i) \exp \int_D d^2z J_a A^a_{\bar{z}} |B\rangle = 0. \tag{26}$$

Call D_i an infinitesimally small disk centered around the i-th puncture. Define $C_i = \partial D_i$ and $\delta_n \mathcal{O}^i(z_i) = \oint_{C_i} \frac{dw}{2\pi i} w^n [J^-(w) - k] \mathcal{O}^i(z_i)$. By deforming the contour of integration C past all the operator insertions, and using the OPE of the affine-Lie algebra, we find

$$\sum_i \langle 0|R \prod_{l\neq i} \mathcal{O}^l(z_l) \delta_n \mathcal{O}^i(z_i) \exp \int_D d^2z J_a A^a_{\bar{z}} |B\rangle + \left[\int_{D-\sum_i D_i} d^2z z^n F^+_{z\bar{z}} + \oint_{C-\sum_i C_i} dz z^n A^+_{\bar{z}} \right] Z_{B\mathcal{O}} = 0. \tag{27}$$

The first term in brackets vanishes because of Gauss law, while the second vanishes on any smooth gauge field configuration obeying the asymptotic condition Eq. (23)[4], so we arrive at the constraint

$$\sum_i \langle 0|R \prod_{l\neq i} \mathcal{O}^l(z_l) \delta_n \mathcal{O}^i(z_i) \exp \int_D d^2z J_a A^a_{\bar{z}} |B\rangle = 0. \tag{28}$$

In the BRST formalism, the constraint translates into

$$\sum_i \langle 0|R \prod_{l\neq i} \mathcal{O}^l(z_l) [Q_B, \mathcal{O}^i(z_i)] \exp \int_D d^2z J_a A^a_{\bar{z}} |B\rangle = 0. \tag{29}$$

In particular, it is satisfied if the operators $\mathcal{O}^i(z_i)$ are BRST-invariant.

[4]Recall that $A_{\bar{z}}$ corresponds to A_- in the euclidean theory.

3 The $sl(2,R)$ Currents

In this Section we show that, given a generic 2-dimensional CFT, it is always possible to add a universal set of auxiliary free fields such that the resulting theory carries an affine $sl(2,R)$ algebra structure. Our construction is inspired by free field realizations of affine-Lie algebras first introduced in [24, 25], further generalized and analyzed in [12, 26, 27, 28, 29, 30, 31] (see [32] for a review and further references). From now on we switch to Euclidean notation and we use holomorphic-antiholomorphic (rather than left-right moving) coordinates.

We start with a CFT (henceforth referred to as the "matter," or "physical" CFT) with stress tensor T_m and central charge c_m, satisfying the OPE

$$T_m(z)T_m(w) \sim \frac{c_m/2}{(z-w)^4} + \frac{2T_m(w)}{(z-w)^2} + \frac{\partial T_m(w)}{z-w}, \tag{30}$$

but otherwise generic. In particular, we do not make any assumption regarding the spectrum of the operators, or the nature of the interactions. We can regard this CFT as the boundary theory, describing some generic matter coupled to gravity in AdS_3. Then, the stress tensor T_m represents the generator of the Virasoro algebra of asymptotic symmetries of the AdS_3 theory. As a first step in the construction, we add the following set of auxiliary free fields: two scalars fields $\rho(z)$ and $\chi(z)$, and a pair of bosonic ghost fields (β,γ) of weight $(0,1)$. The scalar field ρ is a "ghost": its kinetic term has opposite sign compared to that of a physical scalar. These fields have OPEs

$$\beta(z)\gamma(w) \sim \frac{1}{z-w}, \qquad \chi(z)\chi(w) \sim \ln\frac{1}{z-w}, \qquad \rho(z)\rho(w) \sim \ln(z-w). \tag{31}$$

Furthermore, we assume that the field χ has a background charge α_χ, such that the central charge in the χ-sector, $c_\chi \equiv 1 - 12\alpha_\chi^2$, is equal and opposite to that of the physical sector. So, we take for χ a (non-unitary) stress-energy tensor of the form

$$T_\chi = -\frac{1}{2}(\partial\chi)^2 + i\alpha_\chi\partial^2\chi, \tag{32}$$

$$1 - 12\alpha_\chi^2 = -c_m. \tag{33}$$

Next, we define the $sl(2,R)$ currents

$$\begin{aligned} J^+(z) &= -\beta(z)[\gamma(z)]^2 - \alpha_+\gamma(z)\partial\rho(z) + k\partial\gamma(z) + (k+2)[\beta(z)]^{-1}\left[T_m(z) + T_\chi(z)\right], &(34)\\ J^3(z) &= \beta(z)\gamma(z) + \frac{1}{2}\alpha_+\partial\rho(z), &(35)\\ J^-(z) &= \beta(z), &(36)\end{aligned}$$

where $\alpha_+ = \sqrt{2k+4}$. These currents are similar to the ones appearing in the standard free-field realization of the $sl(2,R)$ affine algebra as it appears in [12], the difference being the presence of the last term in $J^+(z)$[5]. As a consequence of the OPEs in Eq. (31), and of the stress-energy tensor OPE, these operators satisfy the $sl(2,R)$ affine-Lie algebra at level k:

$$\begin{aligned} J^+(z)J^-(w) &\sim \frac{2J^3(w)}{z-w} + \frac{k}{(z-w)^2}, \\ J^3(z)J^+(w) &\sim \frac{J^+(w)}{z-w}, \qquad J^3(z)J^-(w) \sim -\frac{J^-(w)}{z-w}, \\ J^3(z)J^3(w) &\sim \frac{k/2}{(z-w)^2}, \qquad J^+(z)J^+(w) \sim 0, \qquad J^-(z)J^-(w) \sim 0. \end{aligned} \tag{37}$$

The coefficient of the last term in J^+ is fixed by demanding that $J^+(z)J^+(w) \sim 0$. This also imposes the requirement that $T_m + T_\chi$ has vanishing central charge. Up to this point, the level k is arbitrary, and not related to the value of the central charge of the physical theory. However, we shall see that, if we require this extended theory to be physically equivalent to the CFT we started with, the value of k will be uniquely determined in terms of c_m.

The standard Sugawara stress tensor associated to the $sl(2,R)$ algebra is

$$\begin{aligned} T^{sug} &= \frac{1}{2(k+2)}\Big(: J^+J^- : + : J^-J^+ : + 2 : J^3J^3 :\Big) \\ &= \beta\partial\gamma + \frac{1}{2}(\partial\rho)^2 + \frac{1}{\alpha_+}\partial^2\rho + T_\chi + T_m, \end{aligned} \tag{38}$$

where $::$ denotes the normal ordering. This stress-energy tensor has central charge $c_{SL_2} = 3k/(k+2)$. Under T^{sug}, the currents in Eq. (34) are primary operators of weight one. Notice that the field ρ has a background charge $1/\alpha_+$, so, its contribution to the central charge is $c_\rho = 1 - 12\alpha_+^{-2} = (k-4)/(k+2)$. In the next Section, we shall see that, to be able to impose the constraints discussed in Section 2, we will have to change the conformal weight of J^-, by adding an improvement term to T^{sug}.

[5]The presence of an inverse power of the ghost field $\beta(z)$ in the definition of J^+ may seem unusual, and one may worry that it is ill-defined. In general, one can define an arbitrary power of a field through its OPE with other fields, as is done in the context of "fractional calculus" (see e.g. [33] and references therein). Alternatively, one can "bosonize" the (β,γ) pair by trading it for two scalar fields $\phi(z)$, $\psi(z)$, with OPEs $\phi(z)\phi(w) \sim -\ln(z-w)$, $\psi(z)\psi(w) \sim \ln(z-w)$, and the identifications

$$\beta = \exp(\phi - \psi), \qquad \gamma = \partial\psi \exp(\psi - \phi).$$

Then one has $\beta^{-1} \equiv \exp(\psi - \phi)$.

4 Constraints

As we have seen in Section 2, imposing AdS_3 boundary conditions is equivalent to imposing appropriate constraints on the $sl(2,R)$ currents, since the asymptotic isometries of AdS_3 generate only the Virasoro algebra, rather than the full affine-Lie algebra. This is true for pure gravity as well as for gravity coupled to matter, as long as the matter fields have a boundary behavior that does not spoil the asymptotic form of the metric. To reduce the full $sl(2,R)$ symmetry to the Virasoro algebra, we need to impose the constraint Eq. (24), $J^-(z) = k$. Although the reduced theory has only a Virasoro symmetry, this constraint does not remove all auxiliary fields from the spectrum, since it does not act on the fields ρ and χ. To eliminate these extra fields, and to reduce the current algebra to the *physical* Virasoro algebra associated to T_m, we need to impose some additional constraints. It turns out that the extra condition $\partial\rho = \partial\chi$ is sufficient to our purpose.

We are going to impose these constraints in a consistent way, using the BRST formalism, in the next Sections. Here we want to give a heuristic idea of how these constraints, together with a condition on k , give the stress tensor T_m, and the correct central charge c_m, starting from T^{sug} and its central charge c_{SL_2}.

First of all, notice that the constraint $J^-(z) = k$ is meaningless if $J^-(z)$ is a field of dimension one. This is clear from dimensional analysis. Equivalently, this constraint does not commute with the Sugawara Hamiltonian, $L_0^{sug} = (2\pi i)^{-1} \oint dz\, zT^{sug}(z)$:

$$\left[L_0^{sug}, J^-(w)\right] = J^-(w) + w\partial J^-(w). \tag{39}$$

From this equation, we can see that that difficulty is overcome if we modify the stress tensor in such a way that $J^-(z)$ becomes a field of dimension zero. The following *twisted* stress-energy tensor [12] has that property:

$$T^{impr} = T^{sug} - \partial J^3, \qquad\qquad T^{impr}(z)J^-(w) \sim \frac{\partial J^-(w)}{z-w}. \tag{40}$$

This assigns dimension zero, one and two to J^-, J^3, J^+, respectively[6]. In terms of the elementary fields we have:

$$\begin{aligned} T^{impr} &= -\partial\beta\gamma + \frac{1}{2}(\partial\rho)^2 - \alpha_\rho\partial^2\rho + T_\chi + T_m \\ &= T_m + \left[-\partial\beta\gamma + \frac{1}{2}(\partial\rho)^2 - \frac{1}{2}(\partial\chi)^2 - \alpha_\rho\partial^2\rho + i\alpha_\chi\partial^2\chi\right]. \end{aligned} \tag{41}$$

[6]Notice, however, the appearance of a central term of the form $-\frac{3k}{(z-w)^3}$ in the $T^{impr}J^3$ OPE, which makes J^3 a *quasi*-primary field.

We see that w.r.t. T^{impr}, β has dimension zero, and γ has dimension one. The background charge of ρ and the central charge become

$$\alpha_\rho = (k+1)/\sqrt{2k+4}, \qquad c_{impr} = 2 + (1 - 12\alpha_\rho^2). \tag{42}$$

(Recall that, in our construction, $T_\chi + T_m$ contributes zero to the central charge). If we impose the constraints

$$\beta(z) = k, \qquad \partial\rho = \partial\chi, \tag{43}$$

and we fix k such that

$$\alpha_\rho = i\alpha_\chi, \tag{44}$$

the term in brackets in the second line of Eq. (41) disappears, and the improved stress tensor reduces to the physical one, with central charge c_m. At this point, the level k of the $sl(2,R)$ algebra is no longer arbitrary: from Eqs. (33,44) it follows that k is fixed in terms of c_m by

$$\frac{(k+1)^2}{k+2} = -\frac{c_m + 1}{6}. \tag{45}$$

Notice that, if we start from a physical CFT with positive central charge, this relation requires $k+2$ to be negative. This is rather natural: recall for example that, in the case of pure gravity in AdS_3, $k = -l/4G_N$ [15]. In the semi-classical limit, in which k is large, Eq. (45) gives $c_m \simeq -6k = 3l/2G_N$, which agrees with the formula for the "classical" central charge of AdS_3 gravity found long ago by Brown and Henneaux [8].

From the above discussion, it seems reasonable that the constraints Eqs. (43) should be enough to project out all the additional auxiliary fields we have introduced to build the $sl(2,R)$ currents. At this level, to require Eq. (44) sounds rather arbitrary: after all, by imposing a constraint we eliminate some degrees of freedom, independently of the numerical parameters of the theory. In the next Section, we will see that Eq. (44) is essential if we want to impose the constraints consistently, using the BRST method.

5 The BRST Charge

In this Section we show how the constraints can be imposed using the BRST formalism, as it was done in [12] in the free field case. It is useful to introduce the quantity $\alpha_0 = -i\alpha_\rho$, which is real when $k+2$ is negative, and change variables to

$$X^+(z) = \frac{1}{\sqrt{2}}\left[\rho(z) + \chi(z)\right], \qquad X^-(z) = \frac{1}{\sqrt{2}}\left[\rho(z) - \chi(z)\right],$$
$$X^+(z)X^+(w) \sim 0, \qquad X^+(z)X^-(w) \sim \ln(z-w), \qquad X^-(z)X^-(w) \sim 0. \tag{46}$$

In terms of these variables, the second constraint in Eq. (43) reads $\partial X^-(z) = 0$, and the part of T^{impr} depending on scalar fields is

$$T^{impr}[X^+, X^-] = \partial X^+ \partial X^- - i\frac{\alpha_0 + \alpha_\chi}{\sqrt{2}}\partial^2 X^- - i\frac{\alpha_0 - \alpha_\chi}{\sqrt{2}}\partial^2 X^+. \tag{47}$$

Since we have two bosonic constraints we introduce two independent sets of fermionic ghosts (b, c) and (B, C), of conformal weights $(0, 1)$, and $(1, 0)$, respectively. Their OPEs are

$$\begin{aligned} b(z)c(w) &\sim \frac{1}{z-w}, & c(z)b(w) &\sim \frac{1}{z-w}, \\ B(z)C(w) &\sim \frac{1}{z-w}, & C(z)B(w) &\sim \frac{1}{z-w}. \end{aligned} \tag{48}$$

Then, we define the total stress-energy tensor

$$T^{tot}(z) = T^{impr}(z) + \partial b(z)c(z) + \partial C(z)B(z). \tag{49}$$

Each set of ghosts contribute (-2) to the central charge, so from Eq. (42) the theory now has

$$c_{tot} = c_{impr} - 4 = -1 + 12\alpha_0^2. \tag{50}$$

Next, we define the following BRST current and charge:

$$j_B(z) = [\beta(z) - k]c(z) + \partial X^-(z)C(z), \qquad Q_B = \oint \frac{dz}{2\pi i} j_B(z). \tag{51}$$

The first term is the same one that was used in [12] to impose the constraint $J^- = 1$. The charge Q_B is nilpotent, since the two terms anticommute with each other, and have regular OPE with themselves. However, Q_B is *not* conserved, in general: as one can see using Eqs. (46,47,48), the OPE between the stress energy tensor and the BRST current is

$$T^{tot}(z)j_B(w) \sim \frac{j_B(w)}{(z-w)^2} + \frac{\partial j_B(w)}{z-w} + i\sqrt{2}\,(\alpha_0 - \alpha_\chi)\,\frac{C(w)}{(z-w)^3}. \tag{52}$$

So, Q_B does not commute with T^{tot}; we find instead

$$\left[T^{tot}(z), Q_B\right] = -\frac{i}{\sqrt{2}}\,(\alpha_0 - \alpha_\chi)\,\partial^2 C(z). \tag{53}$$

Therefore, requiring that Q_B is conserved forces us to impose $\alpha_0 = \alpha_\chi$[7].

[7] One may ask what happens, if we try to impose a more general linear constraint, involving X^+ and X^-, with a term in j_B of the form $(a\partial X^+ + b\partial X^-)C$. It is easy to see that the nilpotency of Q_B requires that either a or b vanishes, and then its conservation implies that $\alpha_0 = \pm\alpha_\chi$. The two choices yield the same physical spectrum.

With this definition of the BRST charge, the total stress tensor of the theory is physically equivalent to that of the original CFT, T_m. As a first consequence of Eqs. (50,33), and since $\alpha_0 = \alpha_\chi$, we have $c_{tot} = c_m$. Moreover, $T^{tot} = T_m$ modulo a BRST-exact operator:

$$\begin{aligned} T^{tot} - T_m &= \partial X^+ \partial X^- - i\frac{2\alpha_0}{\sqrt{2}}\partial^2 X^- - \partial\beta\,\gamma + \partial b\, c + \partial C\, B \\ &= -\left\{ Q_B,\, \gamma\,\partial b + i\sqrt{2}\alpha_0 \partial B - (\partial X^+) B \right\}. \end{aligned} \tag{54}$$

This shows that the physical CFT, and the extended theory with $sl(2,R)$ symmetry after BRST-projection, share the same stress tensor. In the next Section we will show that they also have the same spectrum of physical states.

6 BRST Cohomology

The BRST operator defined in the previous Section defines a cohomology on the Hilbert space of the full $sl(2,R)$ theory. In this Section we show that this cohomology is isomorphic to the space of states of the physical CFT.

The Hilbert space of the $sl(2,R)$ theory (henceforth referred to as $\mathcal{H}_{SL_2}$) is the tensor product of the original CFT Hilbert space $\mathcal{H}_m$ with the free-field Fock spaces of the fields χ and ρ, and of the pairs (β,γ), (b,c), and (B,C):

$$\mathcal{H}_{SL_2} = \mathcal{H}_m \otimes \mathcal{H}_{\beta,\gamma} \otimes \mathcal{H}_{b,c} \otimes \mathcal{H}_\rho \otimes \mathcal{H}_\chi \otimes \mathcal{H}_{B,C}. \tag{55}$$

The space of physical states $\mathcal{H}_{phys}$ is defined as the Q_B-cohomology on $\mathcal{H}_{SL_2}$, referred to as $H(Q_B)$. This is the space of states annihilated by Q_B, modulo exact states, i.e. states belonging to the image of Q_B. The BRST operator is the sum of two terms, each of which acts on different factors in the tensor product in Eq. (55)

$$\begin{aligned} Q_B &= \hat{Q}_1 + \hat{Q}_2 = Q_1 \otimes 1_2 + (-)^{F_1} \otimes Q_2, \\ Q_1 &= \oint \frac{dz}{2\pi i}[\beta(z) - k]c(z), \qquad Q_2 = \oint \frac{dz}{2\pi i}\partial X^-(z) C(z). \end{aligned} \tag{56}$$

Q_1 acts nontrivially only on $\mathcal{H}_1 \equiv \mathcal{H}_{\beta,\gamma} \otimes \mathcal{H}_{b,c}$, and as the identity on the rest. Q_2 acts only on $\mathcal{H}_2 \equiv \mathcal{H}_\rho \otimes \mathcal{H}_\chi \otimes \mathcal{H}_{B,C}$. $(-)^{F_1}$ denotes the fermion parity on $\mathcal{H}_1$. Clearly $\hat{Q}_1$ and $\hat{Q}_2$ anticommute, and are separately nilpotent. From a general cohomology-theoretical result, it follows that, in this situation, the total cohomology is the tensor product of the two:

$$H(\hat{Q}_1 + \hat{Q}_2, \mathcal{H}_1 \otimes \mathcal{H}_2) = H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2). \tag{57}$$

This statement is known in algebraic geometry as Künneth's formula (see e.g. [34]). For the benefit of the reader, we present a proof adapted to our case in Appendix B. Clearly, since both Q_1 and Q_2 act as the identity operator on $\mathcal{H}_m$, the latter will be part of the cohomology as a separate factor. Therefore the space of physical states has the form

$$\mathcal{H}_{phys} = \mathcal{H}_m \otimes H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2). \tag{58}$$

In what follows we show that both $H(Q_1) \equiv H(Q_1, \mathcal{H}_1)$ and $H(Q_2) \equiv H(Q_2, \mathcal{H}_2)$ are essentially one-dimensional, so that the spectrum of the $sl(2, R)$ theory after BRST-projection reduces to the spectrum of the original CFT.

6.1 Q_1 Cohomology

Consider the Hilbert space $\mathcal{H}_1$. This is the Fock space of oscillators $(\beta_n, \gamma_n, b_n, c_n)$, defined by the expansions

$$\begin{aligned} \beta(z) &= \sum_n \frac{\beta_n}{z^n}, & \gamma(z) &= \sum_n \frac{\gamma_n}{z^{n+1}}, \\ b(z) &= \sum_n \frac{b_n}{z^n}, & c(z) &= \sum_n \frac{c_n}{z^{n+1}}, \end{aligned} \tag{59}$$

with non-vanishing (anti)commutation relations

$$[\beta_n, \gamma_m] = \delta_{n+m,0}, \qquad \{c_n, b_m\} = \delta_{n+m,0}. \tag{60}$$

The space $\mathcal{H}_1$ contains a vacuum state $|0\rangle \equiv |0\rangle_{\beta,g} \otimes |0\rangle_{b,c}$ satisfying

$$\begin{aligned} \beta_n|0\rangle = b_n|0\rangle &= 0, & n &\geq 0, \\ \gamma_n|0\rangle = c_n|0\rangle &= 0, & n &\geq 1. \end{aligned} \tag{61}$$

All other states are built by repeatedly applying the remaining operators on the vacuum.
In terms of oscillators, the BRST operator Q_1 reads

$$Q_1 = \sum_n (\beta_n - k\,\delta_{n,0})\, c_{-n}. \tag{62}$$

Physical states are Q_1-closed, $(Q_1|\Phi\rangle = 0)$, with two states $|\Phi\rangle$ and $|\Phi'\rangle$ being equivalent if their difference is Q_1-exact, $(|\Phi\rangle = |\Phi'\rangle + Q_1|\Psi\rangle$ for some $|\Psi\rangle)$. The vacuum state is not closed, since $Q_1|0\rangle = (\beta_0 - k)c_0|0\rangle = -k\, c_0|0\rangle$. To get a closed state we must apply some γ_0 oscillators. This

does not change the energy, since the Virasoro operator L_0 does not contain the zero modes β_0 and γ_0 [see Eq. (41)]:

$$L_0 = \sum_n n : \beta_{-n}\gamma_n : + ..., \tag{63}$$

In particular, we can take as the physical vacuum in the $(\beta\gamma)$ sector the closed state $e^{k\,\gamma_0}|0\rangle$. It turns out that this vacuum state is the only nontrivial state in the Q_1 cohomology. This is because [12] the pairs $(\beta - k, \gamma)$ and (b, c) constitute a Kugo-Ojima quartet [8] [35], which is always projected out of the physical Hilbert space $H(Q_1)$. To see this in our case, one recursively constructs the following set of projection operators:

$$\begin{aligned} P^{(0)} &= \exp(k\,\gamma_0)|0\rangle\langle 0|, \\ P^{(N)} &= \frac{1}{N}\sum_{n\geq 1}\left(b_{-n}P^{(N-1)}c_n - \beta_{-n}P^{(N-1)}\gamma_n\right) \\ &+ \frac{1}{N}\sum_{n\geq 0}\left[c_{-n}P^{(N-1)}b_n + \gamma_{-n}P^{(N-1)}\left(\beta_n - k\,\delta_{n,0}\right)\right]. \end{aligned} \tag{64}$$

The operators $P^{(N)}$ project on subspaces containing N excitations of the modes of β, γ, b, c. They commute with Q_1, and constitute a complete set on $Ker\, Q_1$: $\sum_N P^{(N)} = \mathbf{1}_{Ker\, Q_1}$. Moreover, for $N \geq 1$ they are Q_1-exact:

$$\begin{aligned} P^{(N)} &= \left\{Q_1, R^{(N)}\right\}, \\ R^{(N)} &= -\frac{1}{N}\sum_{n\geq 1} b_{-n}P^{(N-1)}\gamma_n + \frac{1}{N}\sum_{n\geq 0}\gamma_{-n}P^{(N-1)}b_n. \end{aligned} \tag{65}$$

Therefore, any closed state $|\Psi\rangle$ can be written as

$$\begin{aligned} |\Psi\rangle &= \sum_{N\geq 0} P^{(N)}|\Psi\rangle = P^{(0)}|\Psi\rangle + \sum_{N\geq 1}\left\{Q_1, R^{(N)}\right\}|\Psi\rangle \\ &= \exp(k\,\gamma_0)|0\rangle\langle 0|\Psi\rangle + Q_1\left(\sum_{N\geq 1} R^{(N)}|\Psi\rangle\right). \end{aligned} \tag{66}$$

So, the Q_1 cohomology reduces to the one-dimensional subspace generated by $\exp(k\gamma_0)|0\rangle$, i.e. it contains only the vacuum state:

$$H(Q_1) = \{\exp(k\,\gamma_0)|0\rangle_{\beta,\gamma} \otimes |0\rangle_{b,c}\}. \tag{67}$$

[8] A Kugo-Ojima quartet is a set of four fields that realize the following representation of the BRST algebra:

$$[Q, b\} = \beta, \qquad [Q, \beta\} = 0, \qquad [Q, \gamma\} = c, \qquad [Q, c\} = 0.$$

6.2 Q_2 Cohomology

A similar discussion applies to Q_2. Consider now the Hilbert space $\mathcal{H}_2$. This is the Fock space of the oscillators

$$(x^+, x^-, a_n^+, a_n^-, B_n, C_n),$$

defined by the expansions

$$X^+(z) = x^+ + a_0^+ \ln z - \sum_{n\neq 0} \frac{a_n^+}{n} z^{-n}, \qquad X^-(z) = x^- + a_0^- \ln z - \sum_{n\neq 0} \frac{a_n^-}{n} z^{-n}, \tag{68}$$

$$B(z) = \sum_n \frac{B_n}{z^{n+1}}, \qquad C(z) = \sum_n \frac{C_n}{z^n}, \tag{69}$$

with non-vanishing (anti)commutation relations

$$\begin{aligned} \left[a_n^+, a_m^-\right] &= n\delta_{n+m,0}, \\ \left[a_n^+, x^-\right] &= \delta_{n,0}, \qquad \left[a_n^-, x^+\right] = \delta_{n,0}, \\ \{C_n, B_m\} &= \delta_{n+m,0}. \end{aligned} \tag{70}$$

The space $\mathcal{H}_2$ contains a vacuum state $|0\rangle \equiv |0\rangle_{+,-} \otimes |0\rangle_{B,C}$ annihilated by $a_n^\pm$, $n \geq 0$, by C_n, B_n, $n > 0$ and by B_0. All other states are built by repeatedly applying the remaining operators on the vacuum.

In terms of oscillators, the BRST operator Q_2 reads

$$Q_2 = \sum_n a_n^- C_{-n}. \tag{71}$$

The Q_2-cohomology is found with the same argument we used in the previous Subsection for the Q_1-cohomology. The result is essentially the same, up to the presence of the zero mode of the field x^-. Indeed, the pairs of fields $(\partial X^-, \partial X^+)$ and (B, C) constitute another Kugo-Ojima quartet and, therefore, states created by their modes are projected out of the cohomology. This time the vacuum state is closed, so the projector $P^{(0)}$ is just the projector on the vacuum of $\mathcal{H}_2$. This leaves only the zero modes of x^+ and x^- as possible candidates to produce other physical states, besides the vacuum. Clearly any state of the form $f(x^+, x^-)|0\rangle$ is unphysical unless f is independent of x^+, since $[Q_2, f(x^+, x^-)] = C_0[\partial f(x^+, x^-)/\partial x^+]$. However, there is no constraint on the x^- dependence. We can work with states with definite a_0^+ and a_0^- eigenvalues, of the form $|p^+, p^-\rangle = \exp(p^+ x^- + p^- x^+)|0\rangle$. Physical states are required to have $p^- = 0$. All the states $|p^+, 0\rangle$, with arbitrary p^+, are closed, but they are not exact, and they are not even

BRST-equivalent to the zero-charge vacuum, since they are orthogonal to it. However, they are all degenerate in energy with the vacuum. Indeed, from Eq. (47) we have

$$L_0^{tot}|p^+,0\rangle = \left(a_0^+ a_0^- + \text{ terms commuting with } x^-\right)\exp(p^+x^-)|0\rangle = p^+a_0^-|p^+,0\rangle = 0. \tag{72}$$

So, we can arbitrarily choose any one of these states as "the vacuum," and all matrix element will be independent of this choice.

From the result of the last two Subsections, it follows that

$$H(Q_B) \simeq \mathcal{H}_m \otimes \exp(k\gamma_0)|0\rangle_{aux}, \tag{73}$$

where $|0\rangle_{aux}$ is the vacuum of the auxiliary fields we introduced in Section 3. This implies that the physical Hilbert space $H(Q_B)$ can be identified with the Hilbert space $\mathcal{H}_m$ of the original CFT.

7 Irreducible Representations

In this Section we show how one can explicitly construct irreducible representations of the $sl(2,R)$ affine Lie algebra generated by the currents in Eq. (34), starting from a primary field of the "matter" CFT. We focus on lowest weight representations of the current algebra, although similar results hold for other types of representations.

A lowest weight, irreducible representation is realized in terms of a set of *affine primary* fields $H_{j,m}(z)$, $(m = j, j+1\ldots)$ that obey

$$J^+(z)H_{j,m}(w) \sim -(m+j)\frac{H_{j,m+1}(w)}{z-w}, \tag{74}$$

$$J^3(z)H_{j,m}(w) \sim m\,\frac{H_{j,m}(w)}{z-w}, \tag{75}$$

$$J^-(z)H_{j,m}(w) \sim (m-j)\frac{H_{j,m-1}(w)}{z-w}. \tag{76}$$

Here the operator $H_{j,j}$ is the lowest weight operator. Each of these operators is associated to a primary state, defined in the usual way as $|j,m\rangle = H_{j,m}(0)|0\rangle$. If we expand the currents in modes,

$$J^a(z) = \sum_n \frac{J^a_n}{z^{n+1}}, \tag{77}$$

these states are annihilated by all positive modes, and form an irreducible representation of the global $sl(2,R)$ subalgebra generated by the zero-modes of the currents. The other states of

the representation of the affine algebra (the affine descendants) are obtained applying negative modes of the currents.

It turns out that for each *Virasoro* primary $\mathcal{O}_h$ of the matter theory with stress energy tensor T_m, one can construct one and only one such irreducible representation.

Let us first describe how representations look like in the free field case, in which the term $T_m + T_\chi$ in J^+ is absent (see e.g. ref. [27]). Clearly, a lowest weight operator of charge j under J^3 is given by

$$H_{j,j}(z) = \exp\left[\frac{2}{\alpha_+}\, j\rho(z)\right]. \tag{78}$$

This satisfies Eqs. (75) and (76) with $m = j$. Now, commute repeatedly this operator with J^+, and use the OPE to read off the other operators of the representation, from the right hand side of Eq. (74). The resulting primary fields are

$$H_{j,m}(z) = [\gamma(z)]^{m-j}\, \exp\left[\frac{2}{\alpha_+} j\rho(z)\right], \qquad m = j, j+1, \ldots \tag{79}$$

This procedure can be carried out in the general case in which the currents are given by Eq. (34). To see how, let us start with a primary operator of weight $\hat{h}$ under $T = T_m + T_\chi$: $\mathcal{O}_{\hat{h}}(z)$. Define

$$H^{(\hat{h})}_{j,j}(z) = V^{(\rho)}_j(z)\mathcal{O}_{\hat{h}}(z), \qquad V^{(\rho)}_j(z) = \exp\left[\frac{2}{\alpha_+}\, j\rho(z)\right]. \tag{80}$$

Clearly, this operator has vanishing OPE with J^-, and has charge j under J^3, so it is a good candidate for a lowest weight operator. On the other hand, its OPE with J^+ reads:

$$\begin{aligned} J^+(z)H^{(\hat{h})}_{j,j}(w) \;&\sim\; -2j\frac{\gamma(w)V^{(\rho)}_j(w)\mathcal{O}_{\hat{h}}(w)}{z-w} \\ &+\; (k+2)\beta^{-1}(z)V^{(\rho)}_j(w)\left[\hat{h}\,\frac{\mathcal{O}_{\hat{h}}(w)}{(z-w)^2} + \frac{\partial\mathcal{O}_{\hat{h}}(w)}{z-w}\right]. \end{aligned} \tag{81}$$

This OPE has a second order pole *unless* $\hat{h} = 0$. So, in order for $H^{(\hat{h})}_{j,j}$ to be an affine primary field, the operator $\mathcal{O}_{\hat{h}}$ must have zero weight. This can be achieved if we recall that $\mathcal{O}_{\hat{h}}$ is a primary not only under T_m, but also under $T_m + T_\chi$. Now, given any primary operator of the *matter* CFT, $\mathcal{O}_h$, of weight h under T_m, we can always build another one, of weight $\hat{h} = 0$ under $T_m + T_\chi$, by dressing $\mathcal{O}_h$ with an appropriate vertex operator involving $\chi(z)$:

$$\mathcal{O}_0 = V^{(\chi)}_q(z)\mathcal{O}_h, \qquad V^{(\chi)}_q(z) \equiv \exp[iq\chi(z)], \tag{82}$$

with q satisfying

$$q(\frac{q}{2} - \alpha_\chi) + h = 0. \tag{83}$$

Therefore, given any primary operator of arbitrary weight h in the matter theory, we can build a primary lowest weight operator for the affine-Lie algebra,

$$H_{j,j}(w) = V_j^{(\rho)}(z) V_q^{(\chi)}(z) \mathcal{O}_h(z), \tag{84}$$

with q and h satisfying the relation Eq. (83), and j arbitrary. The next member of the representation, with $m = j+1$, can be read off from the right hand side of the OPE $J^+(z)H_{j,j}(w)$, and so on for all $m = j+2, j+3, \ldots$. For this procedure to work, one has to check at every step that one does indeed obtain an affine *primary* operator on the r.h.s. of the OPE with J^+, i.e. that every new operator one generates has only first order poles in its OPE with the currents. We argue that this is the case as follows: Assume that the lowest weight $H_{j,j}(z)$ is a primary. We can associate to it a primary *state* in the usual manner, by defining $|j,j\rangle \equiv H_{j,j}(0)|0\rangle$. Since this state is primary, it is annihilated by all modes with n strictly positive, in the mode expansion of the currents: $J^a(z) = \sum J^a_n z^{-n-1}$. Now, define the state $|j, j+1\rangle = J_0^+|j,j\rangle$. This is also a primary state, as one can check using the affine-Lie algebra commutation relations written in terms of modes. We can associate to this primary state an operator $H'(z)$, which is also a primary, and by construction, is precisely the operator $H_{j,j+1}(w)$ appearing on the r.h.s. of the OPE $J^+(z)H_{j,j}(w)$. By applying this argument to $H_{j,j+1}$, which now we know is a primary, we conclude that $H_{j,j+2}$ is a primary, and so on. Therefore, the only nontrivial requirement is, that the lowest weight $H_{j,j}(z)$ is an affine primary.

As a check, we show explicitly that the second operator in the ladder, $H_{j,j+1}$ is indeed a primary, under the only assumptions that $H_{j,j}$ is a primary. We can read off what the operator obtained by acting on $H_{j,j}$ with $J^+(z)$ is, from the residue of first order pole in Eq. (81):

$$H_{j,j+1}(z) = -\gamma V_j^{(\rho)} V_q^{(\chi)} \mathcal{O}_h + \frac{k+2}{2j} \beta^{-1} V_j^{(\rho)} \, \partial \left(V_q^{(\chi)} \mathcal{O}_h \right). \tag{85}$$

This clearly satisfies both Eqs. (75) and (76), with $m = j+1$. The nontrivial part of the proof here, is to check that the OPE with J^+ has no poles of order greater than one. A straightforward calculation shows that this is indeed the case.

We have just seen that we can associate irreducible representations of “angular momentum” j of the current algebra to each Virasoro primary state. Up to now, there are no constraints on the value of j, since it does not depend on h in any way. However, if we take into account the results

of the previous Section, we see that, for a generic value of j, an irreducible representation of the form described above will not contain any physical states. Indeed, because of the Q_2-cohomology conditions, these are constrained to have equal charge under $\partial\rho$ and $\partial\chi$. More explicitly, it is apparent from the above construction, that a generic primary state in a representation of type j has the form

$$|\Psi\rangle = H \exp\left[\frac{2}{\alpha_+} j\rho(0)\right] \exp\left[iq\chi(0)\right] |0\rangle, \tag{86}$$

where the operator H does not contain exponentials of ρ and χ. Section 6.2 taught us that, in order to be physical, the above state must be proportional to $\exp(p^+x^-)|0\rangle = \exp[p^+(\rho - \chi)/\sqrt{2}]|0\rangle$, for some p^+. This implies a relation between j and q:[9]

$$j = \frac{\alpha_+ p^+}{2\sqrt{2}}, \qquad\qquad q = \frac{ip^+}{\sqrt{2}}, \tag{87}$$

or $q = 2j/|\alpha_+|$. Using the relation between q and h given in Eq. (83), we see that a representation with "angular momentum" j, constructed over a matter primary of weight h, survives the BRST projection iff

$$j(h) = \frac{|k+1|}{2}\left(1 - \sqrt{1 - 4h\frac{|k+2|}{(k+1)^2}}\right). \tag{88}$$

Here, we used the relation $\alpha_\chi = \alpha_0 = |k+1|/|\alpha_+|$, and we took the smallest root of the quadratic equation for j. We made this choice since it is natural to associate the primary field with $h = 0$ –the identity operator in the matter CFT– with the trivial representation of $sl(2, R)$, which has $j = 0$. No problem arises in restricting the cohomology to states which satisfy Eq. (88). Indeed, there are two conserved charges in our model, which commute with Q_B, namely, L_0 and a_0^+. The latter is BRST-equivalent to a multiple of the j-charge. Therefore, we can always restrict the cohomology to a subspace of the full Hilbert space, on which these two charges obey some relation, e.g. Eq. (88). With this choice, every irreducible representation of the Virasoro algebra of the matter CFT, with lowest weight h, can be promoted to the (unique) representation of the affine-Lie algebra, with lowest weight $j(h)$, given by Eq. (88). The primary state $|h\rangle \in \mathcal{H}_m$ is identified with the physical state $|\Psi(h)\rangle \in \mathcal{H}_{SL_2}$ given by:

$$\begin{aligned} |\Psi(h)\rangle &= |h\rangle_m \otimes |p^+(h), p^- = 0\rangle_{+,-} \otimes (\exp k\,\gamma_0)|0\rangle_{\beta,\gamma} \otimes |0\rangle_{B,C,b,c}\ , \\ p^+(h) &= -i2(|k+2|)^{-1/2} j(h). \end{aligned} \tag{89}$$

[9]Recall that in our notation $\alpha_+ = \sqrt{2k+4}$ is purely imaginary.

This is the converse of the result of [12], in which it was shown that every irreducible representation of the affine Lie algebra is BRST-projected onto a single irreducible representation of the Virasoro algebra in the physical Hilbert space. Notice that we do not claim that all affine descendants of $|h\rangle$ are physical; quite the opposite: generically, only its *Virasoro* descendants are physical. One may wonder if Eq. (88) implies an upper bound on the possible values of h, which would result in an "exclusion principle" similar to the one found in [6]. This is true if we require $j(h)$ to be real. However, this is not necessary: in a lowest weight representation, j has to be real only if we restrict our attention to *unitary* representations of the global $sl(2, R)$ algebra, generated by the zero modes of the currents. For a compact Lie group this, plus a condition on the weight (integrability), implies that the representation of the full current algebra is unitary. Here, we need not impose such a restriction, since we only demand unitarity (i.e. positivity of the Hilbert space) in the physical, BRST-reduced theory[10]. This only requires that $h > 0$, without any upper bound.

Eq. (88) may seem rather obscure, but its meaning becomes more transparent in the semi-classical limit: when k is large, it reduces to $j \simeq h$. This is what we expect because we are dealing with a model that has two global $sl(2, R)$ structures, one generated by the zero modes of the affine currents, the other by the Virasoro generators L_1, L_0, L_{-1}. While the latter has a clear semi-classical meaning in terms of the gravitational interpretation of the model –it generates the isometries of AdS_3– the former does not. Indeed, it must be projected out of the physical space to give the theory a chance of possessing a metric formulation[11]. Consider, however, the case in which some non-dynamical point-like sources are turned on inside AdS_3. Their energy fixes the value of the Casimir operator of the Virasoro algebra of asymptotic isometries: $1/2(L_1 L_{-1} + L_{-1} L_1 - 2L_0^2) = -h(h-1)$. On the other hand, we can interpret the same point source as a puncture in the disk, with an associated holonomy of the gauge field. As discussed in [11], this gives rise to a representation of the $sl(2, R)$ current algebra based on a lowest weight representation of the global $sl(2, R)$, with weight h. Therefore, at least for this kind of configurations, we do have $j = h$.

[10]Indeed, although the global $sl(2, R)$ algebra admits (infinite-dimensional) unitary representations, the full $sl(2, R)$ current algebra does not [36]. Its representations contain at most a physical positive-definite subspace.

[11]The actions of CS theory and the Einstein-Hilbert action coincide, yet, it is not true that *every* gauge field configuration corresponds to a reasonable classical space-time: for this to be true the dreibein must be invertible. Therefore, it is reasonable to expect that, without additional boundary conditions, as the ones described in Section (2), there may not be a (semi-classical) gravitational interpretation of the model.

8 Conclusions

In this paper, we have found that all 2-d CFTs possess a hidden $sl(2,R)$ affine symmetry. This hidden symmetry is realized by embedding the CFT into a new CFT, which contains more degrees of freedom and more states. It also contains a physical subspace, defined by a BRST cohomology, where it coincides with the original theory.

One aim of our investigation was to extend the "hyperholographic" correspondence between 3-d pure gravity and 2-d Liouville theory to more general CFTs. This may allow for a non-perturbative definition of the Wheeler-De Witt wave function in any consistent 3-d quantum gravity coupled to matter. Before achieving this goal, we should be able to find a wave function that obeys an acceptable Wheeler-De Witt equation. The problem is that Eq. (12), and all its variations, do not look as yet physically acceptable. Specifically, the Gauss law one obtains,

$$F^a_{z\bar{z}}(z,\bar{z})\Psi = \rho^a(z,\bar{z})\Psi, \tag{90}$$

has the correct dependence on the CS gravity fields, but not on the matter fields. The charge density ρ^a, indeed, is essentially the stress-energy tensor of matter, so it must contain a piece quadratic in the the conjugate momenta of the matter fields. What we obtain by "dressing" CFT operators as in Section 7, instead, is linear in the matter fields and their conjugate momenta.

More modestly, though, our construction may be relevant to another problem of 3-d quantum gravity.

Many, in recent years, have conjectured that the microscopic origin of 3-d black-hole entropy may be understandable in pure 3-d gravity. Evidence for and against this idea can be found e.g. in [37]. The rationale for the conjecture is that the CFT microstates that make up the black hole macrostate may be determined by internal properties of the Liouville theory itself. What we know for certain is that these states cannot be the standard normalizable states of the Liouville theory. A recent proposal to identify these states has been put forward in [38].

If this conjecture holds, and if we can identify the local operators that correspond to black hole microstates, then the quantum wave function of the black hole *is* given by Eqs. (13,29). Now, the operators that appear in Eqs. (13,29) are BRST invariant combinations obtained by appropriately dressing the matter CFT operators. Furthermore, the matter CFT is standard: it is unitary, and it has an $SL(2,C)$ invariant vacuum. So, unlike in the Liouville theory, we expect to encounter no ambiguity in identifying the black hole microstates: they are those of the matter CFT, dressed with our auxiliary ghost fields according to e.g. Eq. (84). So, we expect no ambiguity in using Cardy's formula [39] to compute the asymptotic density of states.

This and other questions raised by this paper are worth of future investigation.

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Appendix A: AdS_3 Boundary Conditions

Here, we review what the AdS_3 asymptotic behavior of the metric means in terms of the CS formulation, and what constraints it imposes on the boundary affine $sl(2,R)$ currents.

The conditions defined in [8] for a metric to be asymptotically AdS_3 read:

$$\begin{aligned} \frac{ds^2}{l^2} &= \Big[\Big(1+O(e^{-2r})\Big)\,dr^2 + \Big(e^{2r}+O(1)\Big)\,dx^+dx^- \\ &+ O(1)(dx^+)^2 + O(1)(dx^-)^2 + O(e^{-2r})drdx^+ + O(e^{-2r})drdx^-\Big]. \end{aligned} \tag{A.1}$$

We parametrize the terms up to $O(e^{-r})$ by three r-independent functions $F(x^+,x^-)$, $L(x^+,x^-)$, $\tilde{L}(x^+,x^-)$ in the following way:

$$\frac{ds^2}{l^2} = dr^2 + \Big(e^{2r}+F\Big)\,dx^+dx^- + L\,(dx^+)^2 + \tilde{L}(dx^-)^2 + O(e^{-2r}). \tag{A.2}$$

This implies that the dreibein 1-forms $\{e^+, e^-, e^3\}$ are

$$\frac{1}{l}e^+ = e^r dx^+ + e^{-r}\left(\frac{1}{2}Fdx^+ + \tilde{L}dx^-\right), \tag{A.3}$$

$$\frac{1}{l}e^- = e^r dx^- + e^{-r}\left(\frac{1}{2}Fdx^- + Ldx^+\right), \tag{A.4}$$

$$\frac{1}{l}e^3 = dr, \tag{A.5}$$

up to terms of $O(e^{-2r})$. Since we have

$$ds^2 = (e^3)^2 + \frac{1}{2}\left(e^+e^- + e^-e^+\right), \tag{A.6}$$

we see that the flat metric used to raise and lower flat indexes is

$$\eta = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/2 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Next, we want to find a set of spin connection 1-forms compatible with the requirement that the torsion vanishes asymptotically. As the three independent components we choose $\omega^{+3}, \omega^{-3}, \omega^{+-}$. We need the vanishing torsion equation,

$$de^a + \omega^a_b \wedge e^b = 0, \tag{A.7}$$

to be true at least up to order $O(e^{-r})$. Requiring the $O(e^r)$ terms to vanish fixes the leading order terms in ω^{ab}:

$$\omega^{+3} = e^r dx^+ + O(e^{-r}), \quad \omega^{-3} = e^r dx^- + O(e^{-r}), \quad \omega^{+-} = O(e^{-r}). \tag{A.8}$$

We can parametrize the $O(e^{-r})$ terms by the functions $f(x^+, x^-)$, $g(x^+, x^-)$, $h(x^+, x^-)$, $k(x^+, x^-)$, $p(x^+, x^-)$, $q(x^+, x^-)$ and the 1-form $\chi(x^+, x^-)$:

$$\omega^{+3} = e^r dx^+ + e^{-r}\left(hdx^+ + kdx^- + pdr\right), \tag{A.9}$$

$$\omega^{-3} = e^r dx^- + e^{-r}\left(fdx^+ + gdx^- + qdr\right), \tag{A.10}$$

$$\omega^{+-} = e^{-r}\chi, \tag{A.11}$$

up to terms of $O(e^{-2r})$. Eqs. (A.7) then read

$$de^+ + \omega^+_3 \wedge e^3 + \omega^+_+ \wedge e^+ = \frac{1}{2}\chi \wedge dx^+ + O(e^{-r}), \tag{A.12}$$

$$de^- + \omega^-_3 \wedge e^3 + \omega^-_- \wedge e^- = -\frac{1}{2}\chi \wedge dx^- + O(e^{-r}), \tag{A.13}$$

$$\begin{aligned} de^3 + \omega^3_+ \wedge e^+ + \omega^3_- \wedge e^- &= -\frac{1}{2}\left(gdx^- \wedge dx^+ + qdr \wedge dx^+\right) \\ &\quad -\frac{1}{2}\left(hdx^+ \wedge dx^- + pdr \wedge dx^-\right) + O(e^{-r}). \end{aligned} \tag{A.14}$$

Requiring the $O(1)$ terms to vanish fixes $\chi = 0$, $p = q = 0$ and $g = h$. Defining $\omega_a = -1/2\, \epsilon_{abc}\omega^{bc}$, we have[12]:

$$\omega^+ = e^r dx^+ + e^{-r}\left(hdx^+ + kdx^-\right) + O(e^{-2r}), \tag{A.15}$$

$$\omega^- = -e^r dx^- - e^{-r}\left(hdx^- + fdx^+\right) + O(e^{-2r}), \tag{A.16}$$

$$\omega^3 = 0 + O(e^{-2r}). \tag{A.17}$$

[12] Remember that $\epsilon_{+-3} = 1/2$, $\eta_{+-} = 1/2$, $\eta^{+-} = 2$, so that $\omega^- = 2\omega_+ = 2\left(-1/2\omega^{-3}\right) = -\omega^{-3}$, $\omega^+ = 2\omega_- = 2\left(1/2\omega^{+3}\right) = \omega^{+3}$.

The $sl(2,R)$ gauge connection 1-forms are defined by the relations

$$A^a = \frac{1}{l}\, e^a \, + \omega^a , \qquad \tilde{A}^a = -\frac{1}{l}\, e^a \, + \omega^a , \tag{A.18}$$

so from Eqs. (A.3) and (A.15) we get

$$\begin{aligned} A^+ &= 2e^r dx^+ + e^{-r}\left[\left(\frac{1}{2}F + h\right) dx^+ + \left(\tilde{L} + k\right) dx^-\right] + O(e^{-2r}), && \text{(A.19)} \\ A^- &= e^{-r}\left[\left(\frac{1}{2}F - h\right) dx^- + (L - f)\, dx^+\right] + O(e^{-2r}), && \text{(A.20)} \\ A^3 &= dr + O(e^{-2r}), && \text{(A.21)} \end{aligned}$$

and

$$\begin{aligned} \tilde{A}^+ &= e^{-r}\left[\left(-\frac{1}{2}F + h\right) dx^+ + \left(-\tilde{L} + k\right) dx^-\right] + O(e^{-2r}), && \text{(A.22)} \\ \tilde{A}^- &= -2e^r dx^- - e^{-r}\left[\left(\frac{1}{2}F + h\right) dx^- + (L + f)\, dx^+\right] + O(e^{-2r}), && \text{(A.23)} \\ \tilde{A}^3 &= -dr + O(e^{-2r}). && \text{(A.24)} \end{aligned}$$

From the above equations we see that we need[13] A_- and $\tilde{A}_+$ to vanish at the boundary as $O(e^{-r})$. This justifies the choice $A_- = \tilde{A}_+ = 0$ as boundary conditions on the gauge fields to make the CS action differentiable.

Next, we fix the gauge to set $A_- = \tilde{A}_+ = 0$ everywhere in the 3-d bulk, and not only on the boundary. This means that we can choose $h = F/2$, $f = -L$, $k = -\tilde{L}$. Then, the connections become

$$\begin{aligned} A^+ &= 2e^r dx^+ + e^{-r} F dx^+ + O(e^{-2r}), && \text{(A.25)} \\ A^- &= e^{-r} 2L dx^+ \, + O(e^{-2r}), && \text{(A.26)} \\ A^3 &= dr + O(e^{-2r}), && \text{(A.27)} \end{aligned}$$

and

$$\begin{aligned} \tilde{A}^+ &= -e^{-r} 2\tilde{L} dx^- + O(e^{-2r}), && \text{(A.28)} \\ \tilde{A}^- &= -2e^r dx^- - e^{-r} F dx^- + O(e^{-2r}), && \text{(A.29)} \\ \tilde{A}^3 &= -dr + O(e^{-2r}). && \text{(A.30)} \end{aligned}$$

[13] From now on the upper index always refers to Lie algebra, the lower one to space-time coordinates.

Now we go to the WZW description to find out what consequences Eqs. (A.25) through (A.30) have on the affine currents. The WZW variables are $SL(2,R)$ group elements $U(x^\mu)$, $\tilde{U}(x^\mu)$ defined by the relations

$$A_r = U\partial_r U^{-1}, \qquad A_\phi = U\partial_\phi U^{-1}, \tag{A.31}$$
$$\tilde{A}_r = \tilde{U}\partial_r \tilde{U}^{-1}, \qquad \tilde{A}_\phi = \tilde{U}\partial_\phi \tilde{U}^{-1}. \tag{A.32}$$

These variables are not well-suited to define the $sl(2,R)$ currents, since when Eqs. (A.25) through (A.30) are satisfied, the $\pm$ components of A and $\tilde{A}$ either vanish or diverge on the boundary. Therefore, we write U and $\tilde{U}$ as

$$U(x^\mu) = \exp\{rt^3\}g(x^\mu), \qquad \tilde{U}(x^\mu) = \exp\{-rt^3\}\tilde{g}(x^\mu). \tag{A.33}$$

and construct the currents with the group elements g and $\tilde{g}$:

$$J^a(x^+,x^-) = \lim_{r\to\infty} k\mathrm{Tr}\,[t^a g\partial_\phi g^{-1}], \qquad \tilde{J}^a(x^+,x^-) = \lim_{r\to\infty} k\mathrm{Tr}\,[t^a \tilde{g}\partial_\phi \tilde{g}^{-1}]. \tag{A.34}$$

They are related to the boundry values of the gauge fields by[14]

$$J^+(x^+,x^-) = \lim_{r\to\infty}\frac{k}{2}e^r(A_+)^-, \qquad \tilde{J}^+(x^+,x^-) = \lim_{r\to\infty}\frac{k}{2}e^{-r}(\tilde{A}_-)^-, \tag{A.35}$$
$$J^-(x^+,x^-) = \lim_{r\to\infty}\frac{k}{2}e^{-r}(A_+)^+, \qquad \tilde{J}^-(x^+,x^-) = \lim_{r\to\infty}\frac{k}{2}e^{r}(\tilde{A}_-)^+, \tag{A.36}$$
$$J^3(x^+,x^-) = \lim_{r\to\infty}\frac{k}{2}(A_+)^3, \qquad \tilde{J}^3(x^+,x^-) = \lim_{r\to\infty}\frac{k}{2}(\tilde{A}_-)^3. \tag{A.37}$$

From the above relations and Eqs. (A.25) through (A.30) we see that the currents in eq. (A.34) are well defined, and that they must satisfy

$$J^- = k, \qquad J^3 = 0; \qquad \tilde{J}^+ = -k, \qquad \tilde{J}^3 = 0. \tag{A.38}$$

On the other hand

$$J^+ = kL(x^+,x^-), \qquad \tilde{J}^- = -k\tilde{L}(x^+,x^-), \tag{A.39}$$

are arbitrary, and constitute the boundary degrees of freedom that survive after enforcing the AdS_3 asymptotic conditions.

[14]We make use of the relations $\exp\{rt^3\}t^+\exp\{-rt^3\} = (\exp\{r\})t^+$, $\exp\{rt^3\}t^-\exp\{-rt^3\} = (\exp\{-r\})t^-$.

Appendix B: Künneth's Formula in BRST Cohomology

Here we prove that the cohomology of $Q = Q_1 + Q_2$ is the direct product of the cohomologies of Q_1 and Q_2.

Consider a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are Hilbert spaces with a $\mathbf{Z}_2$ grading given by a fermion parity. Consider a nilpotent operator Q acting on $\mathcal{H}$, of the form $Q = Q_1 \otimes 1_2 + (-)^{F_1} \otimes Q_2$, where F_1 is the Fermion number in the space $\mathcal{H}_1$, with $Q_1^2 = Q_2^2 = 0$. Then we can consider the spaces $H(Q, \mathcal{H}) = Ker\, Q/Im\, Q$, $H(Q_1, \mathcal{H}_1) = Ker\, Q_1/Im\, Q_1$ and $H(Q_2, \mathcal{H}_2) = Ker\, Q_2/Im\, Q_2$, defined as the spaces of equivalence classes $[\psi_\alpha]$, with $\psi_\alpha \sim \psi'_\alpha$ if $\psi'_\alpha = \psi_\alpha + Q_\alpha \xi$, where the index α refers to any of the three pairs $(\mathcal{H}, Q)$, $(\mathcal{H}_1, Q_1)$, $(\mathcal{H}_2, Q_2)$. If Q_1 and Q_2 are hermitian operators, the cohomology spaces carry a Hilbert space structure inherited from that of $\mathcal{H}$, $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

We are going to prove the following

Künneth's Theorem: The spaces $H(Q, \mathcal{H}_1 \otimes \mathcal{H}_2)$ and $H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2)$ are isomorphic as Hilbert spaces.

We prove this statement by constructing a map between the two spaces and then showing that it is a unitary isomorphism[15]. The linear map in question is defined in a natural way on separable elements of $H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2)$ as

$$\begin{aligned} \mu : H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2) &\longrightarrow H(Q, \mathcal{H}_1 \otimes \mathcal{H}_2) \\ \mu\Big([|\alpha\rangle] \otimes [|\beta\rangle]\Big) &= \Big[|\alpha\rangle \otimes |\beta\rangle\Big]. \end{aligned} \tag{B.1}$$

The action of this map depends only on the equivalence classes $[|\alpha\rangle]$, $[|\beta\rangle]$, not on the representatives $\alpha \in Ker\, Q_1$, $\beta \in Ker\, Q_2$: indeed, if $\alpha' \sim \alpha$, then

$$\begin{aligned} \Big[|\alpha'\rangle \otimes |\beta'\rangle\Big] &= \Big[\left(|\alpha\rangle + Q_1|\xi\rangle\right) \otimes |\beta\rangle\Big] = \\ &= \Big[|\alpha\rangle \otimes |\beta\rangle + Q\,\left(|\xi\rangle \otimes |\beta\rangle\right)\Big] = \Big[|\alpha\rangle \otimes |\beta\rangle\Big]. \end{aligned} \tag{B.2}$$

Therefore μ is well defined as a function of the cohomology. To show that μ is one-to-one and onto, we proceed as follows. Decompose the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ as $\mathcal{H}_i = Ker\, Q_i \oplus N_i$, $i = 1, 2$, where N_i is the orthogonal complement to $Ker\, Q_i$. Introducing the projections π_i defined as

[15] We follow closely the proof given in [34] for the analogous theorem in the context of homology of manifolds.

$$
\begin{aligned}
\pi_i : Ker\, Q_i &\longrightarrow H(Q_i) \\
\pi_i(\psi_i) &= [\psi_i],
\end{aligned}
\tag{B.3}
$$

so that $Ker\, \pi_i \equiv Im\, Q_i$ we can further write $Ker\, Q_i = Ker\, \pi_i \oplus V_i$ where V_i is naturally isomorphic to $Ker\, Q_i / Ker\, \pi_i = H(Q_i)$, through the identification $v_i \leftrightarrow [v_i]$. Therefore we have

$$
\begin{aligned}
\mathcal{H}_i &= Im\, Q_i \oplus V_i \oplus N_i, \\
|\psi_i\rangle &= |\sigma_i\rangle + |\tau_i\rangle + |\nu_i\rangle \qquad \forall\, |\psi_i\rangle \in \mathcal{H}_i,
\end{aligned}
\tag{B.4}
$$

with $|\sigma_i\rangle \in Im\, Q_i$, $|\tau_i\rangle \in V_i \subset Ker\, Q_i$, $|\nu_i\rangle \in N_i$ uniquely specified by ψ_i. Now, consider an element $|\Psi\rangle$ of $Ker\, Q \subset \mathcal{H}_1 \otimes \mathcal{H}_2$. According to the above decomposition, we have

$$
\begin{aligned}
|\Psi\rangle &= \sum_k |\psi_1^k\rangle \otimes |\psi_2^k\rangle = \left(\sum_k |\sigma_1^k\rangle \otimes |\sigma_2^k\rangle + \sum_k |\sigma_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k |\tau_1^k\rangle \otimes |\sigma_2^k\rangle\right) \\
&+ \left(\sum_k |\tau_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k |\tau_1^k\rangle \otimes |\nu_2^k\rangle + \sum_k |\nu_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k |\nu_1^k\rangle \otimes |\nu_2^k\rangle \right) \\
&+ \sum_k |\nu_1^k\rangle \otimes |\sigma_2^k\rangle + \sum_k |\sigma_1^k\rangle \otimes |\nu_2^k\rangle.
\end{aligned}
\tag{B.5}
$$

The terms in the first line are Q-exact:

$$
\begin{aligned}
&\sum_k |\sigma_1^k\rangle \otimes |\sigma_2^k\rangle + \sum_k |\sigma_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k |\tau_1^k\rangle \otimes |\sigma_2^k\rangle = \\
&= Q\left(\sum_k |\tilde{\nu}_1^k\rangle \otimes |\sigma_2^k\rangle + \sum_k |\tilde{\nu}_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k (-)^{F_1} |\tau_1^k\rangle \otimes |\tilde{\nu}_2^k\rangle\right),
\end{aligned}
$$

where $|\sigma_i^k\rangle = Q_i |\tilde{\nu}_i^k\rangle$. Moreover, the last term in Eq. (B.5) can be written as

$$
\sum_k |\sigma_1^k\rangle \otimes |\nu_2^k\rangle = Q\left(|\tilde{\nu}_1^k\rangle \otimes |\nu_2^k\rangle\right) - \sum_k (-)^{F_1} |\tilde{\nu}_1^k\rangle \otimes |\tilde{\sigma}_2^k\rangle,
\tag{B.6}
$$

where $|\tilde{\sigma}_2^k\rangle = Q_2 |\nu_2^k\rangle$. Therefore $|\Psi\rangle$ is Q-equivalent to a $|\Psi'\rangle$, which is given by

$$
|\Psi'\rangle = \sum_k |\tau_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k |\tau_1^k\rangle \otimes |\nu_2^k\rangle + \sum_k |\nu_1^k\rangle \otimes |\tau_2^k\rangle + \sum_k |\nu_1^k\rangle \otimes |\nu_2^k\rangle + \sum_{k'} |\nu_1^{k'}\rangle \otimes |\sigma_2^{k'}\rangle.
\tag{B.7}
$$

When $|\Psi'\rangle$ is in $Ker\, Q$ we have

$$
\begin{aligned}
0 = Q|\Psi'\rangle &= \sum_k (-)^{F_1} |\tau_1^k\rangle \otimes |\tilde{\sigma}_2^k\rangle + \sum_k |\tilde{\sigma}_1^k\rangle \otimes |\tau_2^k\rangle \\
&+ \sum_k |\tilde{\sigma}_1^k\rangle \otimes |\nu_2^k\rangle + \sum_k (-)^{F_1} |\nu_1^k\rangle \otimes |\tilde{\sigma}_2^k\rangle + \sum_{k'} |\tilde{\sigma}_1^{k'}\rangle \otimes |\sigma_2^{k'}\rangle.
\end{aligned}
\tag{B.8}
$$

Since all terms on the r.h.s. are linearly independent, this is possible only if they vanish separately. This in turn implies that all terms in Eq. (B.7) except the first one must vanish. For instance, it cannot happen that $\sum_k |\tau_1^k\rangle \otimes |\nu_2^k\rangle \neq 0$ but $0 = \sum_k |\tau_1^k\rangle \otimes |\tilde{\sigma}_2^k\rangle = Q_2(\sum_k |\tau_1^k\rangle \otimes |\nu_2^k\rangle)$, since by construction $\sum_k |\tau_1^k\rangle \otimes |\nu_2^k\rangle$ is *not* in $Ker\, Q_2$. Therefore, any element in $|\Psi\rangle \in Ker\, Q$ is in the same cohomology class as an element of the form

$$|\Psi'\rangle = \sum_k |\tau_1^k\rangle \otimes |\tau_2^k\rangle, \qquad \text{for some } \tau_i^k \in V_i \subset Ker\, Q_i. \tag{B.9}$$

Moreover, $|\Psi\rangle$ is in $Im\, Q$ if and only if $|\Psi'\rangle = 0$. With this result, it is straightforward to show that the map μ defined in Eq. (B.1) is one-to-one and onto. It is obviously onto, since given $|\Psi\rangle \in Ker\, Q$ we have

$$[|\Psi\rangle] = [|\Psi'\rangle] = \Big[\sum_k |\tau_1^k\rangle \otimes |\tau_2^k\rangle\Big] = \mu \sum_k [|\tau_1^k\rangle] \otimes [|\tau_2^k\rangle], \tag{B.10}$$

for some $\tau_i^k \in Ker\, Q_i$. To show that it is one-to-one, take an element $\sum_k [|\alpha^k\rangle] \otimes [|\beta^k\rangle] \in H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2)$, which is mapped into the zero cohomology class of $H(Q, \mathcal{H})$. If $[|\Psi\rangle] = \mu(\sum_k [|\alpha^k\rangle] \otimes [|\beta^k\rangle]) = \Big[\sum_k |\alpha^k\rangle \otimes |\beta^k\rangle\Big] = 0 \in H(Q, \mathcal{H})$, the representative in Eq. (B.9) vanishes. Moreover, thanks to the decomposition in Eq. (B.4), for every k in $\sum_k |\alpha^k\rangle \otimes |\beta^k\rangle$, either $|\alpha^k\rangle$ is Q_1-exact or $|\beta^k\rangle$ is Q_2-exact, so that $\sum_k [|\alpha^k\rangle] \otimes [|\beta^k\rangle] = 0 \in H(Q_1, \mathcal{H}_1) \otimes H(Q_2, \mathcal{H}_2)$.

Lastly, we show that the map μ is unitary. That is, given the scalar product structures of $H(Q_1)$, $H(Q_2)$ and $H(Q_1 + Q_2)$[16], we have

$$\langle \mu(\zeta), \mu(\zeta')\rangle_{H(Q)} = \langle \zeta, \zeta'\rangle_{H(Q_1)\otimes H(Q_2)}. \tag{B.11}$$

This is straightforward: take $\zeta = \sum_k [\alpha_k] \otimes [\beta_k]$, $\zeta' = \sum_l [\alpha'_l] \otimes [\beta'_l]$, then from the definition of μ

$$\begin{aligned} \langle \mu(\zeta), \mu(\zeta')\rangle &= \langle \Big[\sum_k \alpha_k \otimes \beta_k\Big], \Big[\sum_l \alpha'_l \otimes \beta'_l\Big]\rangle = \langle \sum_k \alpha_k \otimes \beta_k, \sum_l \alpha'_l \otimes \beta'_l\rangle \\ &= \sum_{k,l} \langle \alpha_k, \alpha'_l\rangle \langle \beta_k, \beta'_l\rangle. \end{aligned} \tag{B.12}$$

On the other hand:

$$\begin{aligned} \langle \zeta, \zeta'\rangle &= \langle \sum_k [\alpha_k] \otimes [\beta_k], \sum_l [\alpha'_l] \otimes [\beta'_l]\rangle = \sum_{k,l} \langle [\alpha_k], [\alpha'_l]\rangle \langle [\beta_k], [\beta'_l]\rangle \\ &= \sum_{k,l} \langle \alpha_k, \alpha'_l\rangle \langle \beta_k, \beta'_l\rangle, \end{aligned} \tag{B.13}$$

thus proving that μ is a unitary isomorphism.

[16] These are defined by their value on representatives: $\langle [\psi], [\psi]'\rangle := \langle \psi, \psi'\rangle$, which does not depend on the choice of $\psi \in [\psi]$ and $\psi' \in [\psi]'$ since Q-exact states are orthogonal to $Ker\, Q$.

References

[1] G. 't Hooft, arXiv:gr-qc/9310026.

[2] L. Susskind, J. Math. Phys. **36**, 6377 (1995) [arXiv:hep-th/9409089].

[3] J. D. Bekenstein, Phys. Rev. D **7**, 2333 (1973).

[4] O. Aharony, S. S. Gubser, J. M. Maldacena, H. Ooguri and Y. Oz, Phys. Rept. **323**, 183 (2000) [arXiv:hep-th/9905111].

[5] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998) [Int. J. Theor. Phys. **38**, 1113 (1999)] [arXiv:hep-th/9711200].

[6] J. M. Maldacena and A. Strominger, JHEP **9812**, 005 (1998) [arXiv:hep-th/9804085].

[7] J. de Boer, Nucl. Phys. B **548**, 139 (1999) [arXiv:hep-th/9806104].

[8] J. D. Brown and M. Henneaux, Commun. Math. Phys. **104**, 207 (1986).

[9] J. M. Maldacena, talk at Strings 2002, Cambridge, UK; G. T. Horowitz and J. Maldacena, arXiv:hep-th/0310281.

[10] E. Witten, Commun. Math. Phys. **121**, 351 (1989).

[11] S. Elitzur, G. W. Moore, A. Schwimmer and N. Seiberg, Nucl. Phys. B **326**, 108 (1989); G. W. Moore and N. Seiberg, Phys. Lett. B **220**, 422 (1989).

[12] M. Bershadsky and H. Ooguri, Commun. Math. Phys. **126**, 49 (1989).

[13] M. Bershadsky and H. Ooguri, Phys. Lett. B **229**, 374 (1989).

[14] O. Coussaert, M. Henneaux and P. van Driel, Class. Quant. Grav. **12**, 2961 (1995) [arXiv:gr-qc/9506019].

[15] M. Banados, arXiv:hep-th/9901148.

[16] E. Witten, Nucl. Phys. B **311**, 46 (1988).

[17] A. Achucarro and P. K. Townsend, Phys. Lett. B **180**, 89 (1986).

[18] V. G. Turaev and O. Y. Viro, Topology **31**, 865 (1992).

[19] J. Sonnenschein, Nucl. Phys. B **309**, 752 (1988).

[20] M. Stone, Phys. Rev. Lett. **63**, 731 (1989).

[21] M. Henneaux, L. Maoz and A. Schwimmer, Annals Phys. **282**, 31 (2000) [arXiv:hep-th/9910013].

[22] P. Forgacs, A. Wipf, J. Balog, L. Feher and L. O'Raifeartaigh, Phys. Lett. B **227**, 214 (1989).

[23] M. Henneaux, C. Martinez, R. Troncoso and J. Zanelli, Phys. Rev. D **65**, 104007 (2002) [arXiv:hep-th/0201170].

[24] M. Wakimoto, Commun. Math. Phys. **104**, 605 (1986).

[25] A. B. Zamolodchikov, Montreal lectures, 1988, unpublished.

[26] B. L. Feigin and E. V. Frenkel, in *Physics and mathematics of strings*, ed. L. Brink et al. (World Scientific, Singapore, 1990), 271-316; Lett. Math. Phys. **19**, 307 (1990).

[27] A. Gerasimov, A. Morozov, M. Olshanetsky, A. Marshakov and S. L. Shatashvili, Int. J. Mod. Phys. A **5**, 2495 (1990).

[28] D. Bernard and G. Felder, Commun. Math. Phys. **127**, 145 (1990).

[29] P. Bouwknegt, J. G. McCarthy and K. Pilch, Phys. Lett. B **234**, 297 (1990); Nucl. Phys. B **352**, 139 (1991); P. Bouwknegt, J. G. McCarthy, D. Nemeschansky and K. Pilch, Phys. Lett. B **258**, 127 (1991).

[30] M. Kuwahara, N. Ohta and H. Suzuki, Nucl. Phys. B **340**, 448 (1990); M. Kuwahara, N. Ohta and H. Suzuki, Phys. Lett. B **235**, 57 (1990); N. Ohta and H. Suzuki, Nucl. Phys. B **332**, 146 (1990).

[31] P. Furlan, A. C. Ganchev, R. Paunov and V. B. Petkova, Nucl. Phys. B **394**, 665 (1993) [arXiv:hep-th/9201080].

[32] P. Bouwknegt, J. G. McCarthy and K. Pilch, Prog. Theor. Phys. Suppl. **102**, 67 (1990).

[33] J. L. Petersen, J. Rasmussen and M. Yu, Nucl. Phys. Proc. Suppl. **49**, 27 (1996) [arXiv:hep-th/9512175].

[34] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, Wiley & Sons, 1978.

[35] T. Kugo and I. Ojima, Prog. Theor. Phys. **60**, 1869 (1978).

[36] L. J. Dixon, M. E. Peskin and J. Lykken, Nucl. Phys. B **325**, 329 (1989).

[37] S. Carlip, Rept. Prog. Phys. **64**, 885 (2001) [arXiv:gr-qc/0108040]; Class. Quant. Grav. **15**, 3609 (1998) [arXiv:hep-th/9806026].

[38] Y. Chen, arXiv:hep-th/0310234.

[39] J. L. Cardy, Nucl. Phys. B **270**, 186 (1986).